On Hyper B-algebras

Abstract

In this paper, we introduce the notion of a hyper B-algebra, which is a generalization of a B-algebra, and we investigate some related properties. We also introduce and characterize closed and invertible subhyper B-algebras.

Keywords: hyper B-algebras, subhyper B-algebras, closed, invertible

1 Introduction

The *hyperstructure theory* (also called *multialgebras*) was introduced by Marty (1934) at the 8th Congress of Scandinavian Mathematicians. Let H be a nonempty set and $\mathcal{P}^*(H)$ be the set of all nonempty subsets of H. A *hyperoperation* on H is a map from $H \times H$ into $\mathcal{P}^*(H)$. A set H endowed with a family Γ of hyperoperations is called a *hyperstructure* (or *multivalued algebra*). If Γ is a singleton, that is, $\Gamma = \{f\}$, then the hyperstructure is called a *hypergroupoid*, denoted by (H, f). Let $*$ be a hyperoperation on H and $(x, y) \in H \times H$. Then its image under $*$ is denoted by $x * y$ and is called the *hyperproduct* of x and y. If A and B are nonempty subsets of H, then $A * B$ is given by

$$A * B = \bigcup_{a \in A, b \in B} a * b.$$

We shall use $x * y$ instead of $x * \{y\}$, $\{x\} * y$, or $\{x\} * \{y\}$. When $A \subseteq H$ and $x \in H$, we agree to write $A * x$ instead of $A * \{x\}$. Similarly, we write $x * A$ for $\{x\} * A$. In effect, $A * x = \bigcup_{a \in A} a * x$ and $x * A = \bigcup_{a \in A} x * a$.

The notion of B-algebras was introduced by Neggers & Kim (2002). A *B-algebra* is an algebra $(X; *, 0)$ of type (2, 0) (that is, a nonempty set X with a binary operation $*$ and a constant 0) satisfying the following axioms: (I) $x * x = 0$, (II) $x * 0 = x$, and (III) $(x * y) * z = x * (z * (0 * y))$, for all $x, y, z \in X$. The notions of a *subalgebra* and *normality* of B-algebras were introduced by Neggers & Kim (2002) and some of their properties were established. A nonempty subset N of a B-algebra X is called a *subalgebra* of X if $x * y \in N$ for any x, $y \in N$. It is called *normal* in X if for any $x * y, a * b \in N$ implies $(x * a) * (y * b) \in N$. A normal subset of X is a subalgebra of X. For any $x \in X$, we have (P1) $0 * (0 * x) = x$ (Neggers & Kim, 2002).

2 Hyper B-algebra

In this section, we apply the concept of a hyperstructure to a B-algebra, and thus, we introduce the notion of a hyper B-algebra, which is a generalization of a B-algebra, and we investigate some related properties. We now define a hyper B-algebra.

Definition 2.1 A *hyper B-algebra* is a set H with constant 0 and hyperoperation $\circledast$ satisfying the following axioms for all $x, y, z \in H$:

(HI) $0 \in x \circledast x$,

(HII) $x \circledast H = H = H \circledast x$,

(HIII) $(x \circledast y) \circledast z = x \circledast (z \circledast (0 \circledast y))$.

A hyper B-algebra H with constant 0 and hyperoperation $\circledast$ is denoted by $(H; \circledast, 0)$. Since $\circledast$ is a hyperoperation, $x \circledast y \neq \varnothing$ for all $x, y \in H$.

Example 2.2 $(\mathbb{Z}; \circledast, 0)$, $(\mathbb{Q}; \circledast, 0)$, $(\mathbb{R}; \circledast, 0)$, and $(\mathbb{C}; \circledast, 0)$ are hyper B-algebras, where $\circledast$ is defined by $x \circledast y = \{x - y\}$.

Example 2.3 The following are examples of hyper B-algebras.

1. If H is a nonempty set with constant 0, then $(H; \circledast, 0)$ is a hyper B-algebra, where $\circledast$ is defined by $x \circledast y = H$ for all $x, y \in H$.

2. Let $H = \{0, a, b\}$ be a set with $\circledast$ defined by the following table:

$\circledast$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{0, b\}$	$\{0, a\}$
b	$\{b\}$	$\{0, a\}$	$\{0, a\}$

Then $(H; \circledast, 0)$ is a hyper B-algebra.

3. Let $H = \{0, a, b, c\}$ be a set with $\circledast$ defined by the following table:

$\circledast$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0, a, b\}$	$\{0, a, c\}$
a	$\{0\}$	$\{0\}$	$\{0, a, b\}$	$\{0, a, c\}$
b	$\{0, a, b\}$	$\{0, a, b\}$	$\{0, a, b\}$	$\{b, c\}$
c	$\{0, a, c\}$	$\{0, a, c\}$	$\{b, c\}$	$\{0, a, c\}$

Then $(H; \circledast, 0)$ is a hyper B-algebra.

4. Let $H = \{0, a, b, c, d\}$ be a set with $\circledast$ defined by the following table:

$\circledast$	0	a	b	c	d
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{c, d\}$
a	$\{a\}$	$\{0, a\}$	$\{c, d\}$	$\{b, c, d\}$	$\{b, c, d\}$
b	$\{b\}$	$\{c, d\}$	$\{0, b\}$	$\{a, c, d\}$	$\{a, c, d\}$
c	$\{c\}$	$\{b, c, d\}$	$\{a, c, d\}$	H	H
d	$\{c, d\}$	$\{b, c, d\}$	$\{a, c, d\}$	H	H

Then $(H; \circledast, 0)$ is a hyper B-algebra.

Theorem 2.4 Let $(H; *, 0)$ be a B-algebra and define a hyperoperation " $\circledast$ " on H by $x \circledast y = \{x * y\}$ for all $x, y \in H$. Then $(H; \circledast, 0)$ is a hyper B-algebra.

Proof: Let $x, y, z \in H$. Then $0 \in x \circledast x$ by (I). Clearly, $x \circledast H \subseteq H$ and $H \circledast x \subseteq H$. Suppose that $h \in H$. By (P1), (I), and (III), $h = 0 * (0 * h) = (x * x) * (0 * h) = x * [(0 * h) * (0 * x)] \in x \circledast H$. Thus, $H \subseteq x \circledast H$. Hence, $x \circledast H = H$. Also, by (I), (II), (P1), and (III), $h = h * 0 = h * (x * x) = h * [x * (0 * (0 * x))] = (h * (0 * x)) * x \in H \circledast x$. Thus, $H \subseteq H \circledast x$. Hence, $H \circledast x = H$. By (III), $(x \circledast y) \circledast z = x \circledast (z \circledast (0 \circledast y))$. Therefore, $(H; \circledast, 0)$ is a hyper B-algebra. □

A hyper B-algebra H is called *commutative* if $x \circledast y = y \circledast x$ for all $x, y \in H$. The hyper B-algebras in Example 2.3(1-4) are commutative, while the hyper B-algebras in Example 2.2 are not commutative.

The hyperoperation $\circledast$ leads to two new hyperoperations $\triangleright$ and $\triangleleft$ which are defined as follows: $x \triangleleft y = \{h \in H : x \in h \circledast y\}$, $x \triangleright y = \{h \in H : x \in y \circledast h\}$. By (HII), there is at least one h and one k such that $x \in h \circledast y$ and $x \in y \circledast k$.

Remark 2.5 For all $x, y, z \in H$, we have

i. $x \triangleleft y \neq \varnothing$ and $x \triangleright y \neq \varnothing$,

ii. $z \in x \triangleleft y$ if and only if $y \in x \triangleright z$,

iii. if H is commutative, then $x \triangleleft y = x \triangleright y$.

Lemma 2.6 For all $x, y, z \in H$, we have

i. $(x \triangleleft y) \triangleleft z = x \triangleleft (y \circledast (0 \circledast z))$,

ii. $y \in x \triangleright (x \triangleleft y)$,

iii. $y \in x \triangleleft (x \triangleright y)$.

A nonempty subset K of a hyper B-algebra H is called a *semi-subhyper B-algebra* of H if $x \circledast y \subseteq K$ for all $x, y \in K$. Let H be the hyper B-algebra in Example 2.3(4). Then the sets $\{0\}$, $\{0, a\}$, and $\{0, b\}$ are semi-subhyper B-algebras, while $\{a, b\}$ is not, since $a \circledast a = \{0, a\} \nsubseteq \{a, b\}$. By (HI), $0 \in K$ for any semi-subhyper B-algebra K of H. A semi-subhyper B-algebra K of a hyper B-algebra H is called a *subhyper B-algebra* if $x \circledast K = K \circledast x = K$ for all $x \in K$. The sets $\{0\}$, $\{0, a\}$, and $\{0, b\}$ are subhyper B-algebras, while $\{b\}$ is not, since $b \circledast b = \{0, b\} \neq \{b\}$.

Let $x \in K$. If $k \triangleleft x \subseteq K$ and $k \triangleright x \subseteq K$ for every $k \in K$, then we have $K \subseteq K \circledast x$ and $K \subseteq x \circledast K$. Let K be a semi-subhyper B-algebra of H. If $x \triangleleft y \subseteq K$ and $x \triangleright y \subseteq K$ for all $x, y \in K$, then K is a subhyper B-algebra of H.

Lemma 2.7 Let K be a subhyper B-algebra of H. Then for all $x \in K$, $H \setminus K \subseteq (H \setminus K) \circledast x$ and $H \setminus K \subseteq x \circledast (H \setminus K)$.

Proof: Let $x \in K$. Suppose that $y \in H \setminus K$ such that $y \notin (H \setminus K) \circledast x$. By (HII), $y \in H \circledast x$. Since $y \notin (H \setminus K) \circledast x$, it follows that $y \in K \circledast x$. Hence, $y \in K \circledast x \subseteq K \circledast K = K$, a contradiction. Therefore, $H \setminus K \subseteq (H \setminus K) \circledast x$. Similarly, $H \setminus K \subseteq x \circledast (H \setminus K)$. □

Lemma 2.8 Let $A, B \subseteq H$ and K a subhyper B-algebra of H such that $A \subseteq K$. Then $A \circledast (B \cap K) \subseteq (A \circledast B) \cap K$ and $(B \cap K) \circledast A \subseteq (B \circledast A) \cap K$.

Proof: Let $x \in A \circledast (B \cap K)$. Then $x \in a \circledast y$ for some $a \in A$ and $y \in B \cap K$. Since $y \in B$, $y \in K$, and K is subhyper B-algebra of H such that $A \subseteq K$, we have $a \circledast y \subseteq a \circledast B$ and $a \circledast y \subseteq a \circledast K = K$. Hence, $a \circledast y \subseteq (A \circledast B) \cap K$. Therefore, $x \in (A \circledast B) \cap K$ and so $A \circledast (B \cap K) \subseteq (A \circledast B) \cap K$. Similarly, $(B \cap K) \circledast A \subseteq (B \circledast A) \cap K$. □

3 Closed Subhyper B-algebra

Definition 3.1 A subhyper B-algebra K of a hyper B-algebra H is called *closed from the right* (respectively, *from the left*), if $(x \circledast K) \cap K = \varnothing$ (respectively, $(K \circledast x) \cap K = \varnothing$) holds for every $x \in H \setminus K$. K is called *closed* if it is closed from the right and from the left.

Let H be the hyper B-algebra in Example 2.3(4). Then the sets $\{0\}$, $\{0, a\}$, and $\{0, b\}$ are closed subhyper B-algebras, while $\{0, c\}$ is not, since $(d \circledast \{0, c\}) \cap \{0, c\} = \{0, c\} \neq \varnothing$. Let H be the hyper B-algebra in Example 2.3(3). Then the sets $\{0\}$, $\{0, a, b\}$, and $\{0, a, c\}$ are subhyper B-algebras but not closed.

Theorem 3.2 Let K be a subhyper B-algebra of H. Then K is closed from the right (respectively, from the left) if and only if the relation $(x \circledast K) \cap K \neq \varnothing$ (respectively, $(K \circledast x) \cap K \neq \varnothing$) implies that $x \in K$.

Proof: Let K be closed from the right. Suppose that $(x \circledast K) \cap K \neq \varnothing$. If $x \notin K$, then $(x \circledast K) \cap K = \varnothing$, a contradiction. Hence, $x \in K$. Conversely, suppose that $(x \circledast K) \cap K \neq \varnothing$ implies $x \in K$. This means that $(x \circledast K) \cap K = \varnothing$ for all $x \notin K$. Thus, K is closed from the right. Similary, K is closed from the left if and only if the relation $(K \circledast x) \cap K \neq \varnothing$ implies that $x \in K$. □

Theorem 3.3 Let K be a subhyper B-algebra of H. Then K is closed from the right (respectively, from the left) if and only if $x \lhd y \subseteq K$ (respectively, $x \rhd y \subseteq K$) for all $x, y \in K$.

Proof: Let K be closed from the right and let $x, y \in K$. Then for any $h \in x \lhd y$, we have $x \in h \circledast y$. Hence, $(h \circledast K) \cap K \neq \varnothing$. By Theorem 3.2, $h \in K$. Therefore, $x \lhd y \subseteq K$. Conversely, let $(h \circledast K) \cap K \neq \varnothing$ for some $h \in H$. Then $x \in h \circledast y$ for some $x, y \in K$. Hence, $h \in x \lhd y$. Since $x \lhd y \subseteq K$, $h \in K$. By Theorem 3.2, K is closed from the right. Similarly, K is closed from the left if and only if $x \rhd y \subseteq K$ for all $x, y \in K$. □

Proposition 3.4 If K is a closed subhyper B-algebra of H and $x \in K$, then $K = K \lhd x = x \lhd K = K \rhd x = x \rhd K$.

Proof: Let K be closed and $x \in K$. By Theorem 3.3, $K \lhd x \subseteq K$. Let $y \in K$. Then $y \circledast x \subseteq K$ and so $y \in K \lhd x$. Thus, $K \subseteq K \lhd x$. Therefore, $K = K \lhd x$. Similarly, $K = x \lhd K = K \rhd x = x \rhd K$. □

Theorem 3.5 Let K be a subhyper B-algebra of H. Then K is closed from the right (respectively, from the left) if and only if $(H \setminus K) \circledast x = H \setminus K$ (respectively, $x \circledast (H \setminus K) = H \setminus K$) for all $x \in K$.

Proof: Let $x \in K$. Suppose that K is closed from the right. By Lemma 2.7, $H \setminus K \subseteq (H \setminus K) \circledast x$. Let $z \in (H \setminus K) \circledast x$ such that $z \notin H \setminus K$. Then there exists $y \in H \setminus K$ such that $z \in y \circledast x$, or equivalently, $y \in z \lhd x$. By Theorem 3.3, $y \in K$, a contradiction. Hence, $(H \setminus K) \circledast x \subseteq H \setminus K$. Therefore, $(H \setminus K) \circledast x = H \setminus K$. Conversely, suppose that $(H \setminus K) \circledast x = H \setminus K$ for all $x \in K$. Let $a, b \in K$. Then $(H \setminus K) \circledast b = H \setminus K$ and so $(H \setminus K) \circledast b \cap K = \varnothing$. Hence, $a \notin c \circledast b$ for all $c \in H \setminus K$ or equivalently, $c \notin a \lhd b$ for all $c \in H \setminus K$. Thus, $(a \lhd b) \cap (H \setminus K) = \varnothing$ and so $(a \lhd b) \subseteq K$. By Theorem 3.3, K is closed from the right. Similarly, K is closed from the left if and only if $x \circledast (H \setminus K) = H \setminus K$ for all $x \in K$. □

The following corollary summarizes Theorems 3.2, 3.3, and 3.5.

Corollary 3.6 Let K be a subhyper B-algebra of H. Then the following statements are equivalent:

i. K is closed;

ii. the relation $(x \circledast K) \cap K \neq \varnothing$ implies that $x \in K$, and the relation $(K \circledast x) \cap K \neq \varnothing$ implies that $x \in K$;

iii. $x \triangleleft y \subseteq K$ and $x \triangleright y \subseteq K$ for all $x, y \in K$;

iv. $(H \setminus K) \circledast x = H \setminus K = x \circledast (H \setminus K)$ for all $x \in K$.

Theorem 3.7 Let $A, B \subseteq H$ and K a subhyper B-algebra of H such that $A \subseteq K$. If K is closed from the right (respectively, from the left), then $(B \cap K) \circledast A = (B \circledast A) \cap K$ (respectively, $A \circledast (B \cap K) = (A \circledast B) \cap K$).

Proof: Suppose that K is closed from the right. Then by Lemma 2.8, $(B \cap K) \circledast A \subseteq (B \circledast A) \cap K$. Let $x \in (B \circledast A) \cap K$. Then $x \in b \circledast a$ for some $b \in B$, $a \in A$. Hence, $b \in x \triangleleft a$. By Theorem 3.3, $x \triangleleft a \subseteq K$. Thus, $b \in K$ and so $b \in B \cap K$. Therefore, $x \in (B \cap K) \circledast A$ implying that $(B \circledast A) \cap K \subseteq (B \cap K) \circledast A$. Consequently, $(B \cap K) \circledast A = (B \circledast A) \cap K$. Similarly, if K is closed from the left, then $A \circledast (B \cap K) = (A \circledast B) \cap K$. □

Corollary 3.8 Let $A, B \subseteq H$ and K a closed subhyper B-algebra of H such that $A \subseteq K$. Then $A \circledast (B \cap K) = (A \circledast B) \cap K$ and $(B \cap K) \circledast A = (B \circledast A) \cap K$.

Theorem 3.9 Let $A, B \subseteq H$ and K a subhyper B-algebra of H such that $A \subseteq K$. If K is closed from the right (respectively, from the left), then $(B \triangleleft A) \cap K = (B \cap K) \triangleleft A$ (respectively, $(B \triangleright A) \cap K = (B \cap K) \triangleright A$).

Proof: Let K be closed from the right. Since $B \cap K \subseteq B$, $(B \cap K) \triangleleft A \subseteq B \triangleleft A$. Since $(B \cap K) \subseteq K$, $A \subseteq K$, and K is closed, we have $(B \cap K) \triangleleft A \subseteq K$. Hence, $(B \cap K) \triangleleft A \subseteq (B \triangleleft A) \cap K$. Let $x \in (B \triangleleft A) \cap K$. Then there exist $a \in A$, $b \in B$ such that $x \in b \triangleleft a$ or equivalently, $b \in x \circledast a$. Since $x \circledast a \subseteq K$, $b \in K$ and so $b \in B \cap K$. Thus, $x \in (B \cap K) \triangleleft A$. Therefore, $(B \triangleleft A) \cap K \subseteq (B \cap K) \triangleleft A$ and so $(B \triangleleft A) \cap K = (B \cap K) \triangleleft A$. Similarly, if K is closed from the left, then $(B \triangleright A) \cap K = (B \cap K) \triangleright A$. □

Corollary 3.10 Let $A, B \subseteq H$ and K a closed subhyper B-algebra of H such that $A \subseteq K$. Then $(B \triangleleft A) \cap K = (B \cap K) \triangleleft A$ and $(B \triangleright A) \cap K = (B \cap K) \triangleright A$.

4 Invertible Subhyper B-algebra

Definition 4.1 A subhyper B-algebra K of a hyper B-algebra H is called *right* (respectively, *left*) *invertible*, if $(x \triangleleft y) \cap K \neq \varnothing$ implies $(y \triangleleft x) \cap K \neq \varnothing$ (respectively, $(x \triangleright y) \cap K \neq \varnothing$ implies $(y \triangleright x) \cap K \neq \varnothing$) for all $x, y \in H$. K is called *invertible* if it is both right and left invertible.

Let H be the hyper B-algebra in Example 2.3(4). Then the sets $\{0, a\}$ and $\{0, b\}$ are invertible subhyper B-algebras, while $\{0\}$ is not, since $(c \triangleleft d) \cap \{0\} \neq \varnothing$ but $(d \triangleleft c) \cap \{0\} = \varnothing$.

Theorem 4.2 Let K be a subhyper B-algebra of H. Then K is right (respectively, left) invertible if and only if $x \in K \circledast y$ implies $y \in K \circledast x$ (respectively, $x \in y \circledast K$ implies $y \in x \circledast K$) for all $x, y \in H$.

Proof: Let $x, y \in H$. Suppose that K is right invertible. Suppose further that $x \in K \circledast y$. Then there exists $k_1 \in K$ such that $x \in k_1 \circledast y$. This means that $k_1 \in x \triangleleft y$. Thus, $x \triangleleft y \cap K \neq \varnothing$. By Definition 4.1, $(y \triangleleft x) \cap K \neq \varnothing$. Hence, there exists $k_2 \in K$ such that $k_2 \in y \triangleleft x$. Therefore, $y \in (k_2 \circledast x) \subseteq (K \circledast x)$. Conversely, suppose that $(x \triangleleft y) \cap K \neq \varnothing$. Then there exists $k_3 \in K$ such that $k_3 \in x \triangleleft y$. This means that $x \in (k_3 \circledast y) \subseteq (K \circledast y)$. By assumption, $y \in K \circledast x$. Then there exists $k_4 \in K$ such that $y \in k_4 \circledast y$. Hence, $k_4 \in y \triangleleft x$. Therefore, $(y \triangleleft x) \cap K \neq \varnothing$, that is, K is right invertible. Similarly, K is left invertible if and only if $x \in y \circledast K$ implies $y \in x \circledast K$). □

Corollary 4.3 Let K be a subhyper B-algebra of H. Then K is invertible if and only if $x \in K \circledast y$ implies $y \in K \circledast x$, and $x \in y \circledast K$ implies $y \in x \circledast K$ for all $x, y \in H$.

Theorem 4.4 If K is right (respectively, left) invertible in H, then K is closed from the left (respectively, from the right) in H.

Proof: Suppose that K is left invertible. Let $x, y \in K$ and $a \in x \triangleleft y$. Then $x \in a \circledast y \subseteq a \circledast K$. By Theorem 4.2, $a \in x \circledast K = K$. By Theorem 3.3, K is closed from the right. Similarly, if K is right invertible, then K is closed from the left. □

Corollary 4.5 If K is invertible in H, then K is closed in H.

The converse of Corollary 4.5 need not be true. The closed subhyper B-algebra $\{0\}$ in Example 2.3(4) is not invertible.

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