Dominating Subgroups of a Cayley Graph of the Dihedral Group

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ABSTRACT

This paper presents a complete classification of all proper nontrivial subgroups of the dihedral group which form as dominating sets of the Cayley graph with respect to a minimal inverse-symmetric generating subset not containing the identity.

Keywords: Dihedral Group, Cayley Graph, Dominating Set

INTRODUCTION

Cayley graphs are used as representations of computer networks. One important topic in this area is on *domination*. There are several types of dominating sets that are being studied. Lee (2001) characterized the existence of independent perfect domination sets among Cayley graphs of abelian groups. Kwon and Lee (2014) studied perfect domination sets in Cayley graphs by assuming the existence of a perfect dominating set which is a normal subgroup. Another type of domination which interests mathematicians in this area is on *efficient domination*. Knor and Niepel (2011) made research on efficient open domination in digraphs. Some papers revealed that efficient dominating sets of Cayley graphs appear as cosets of subgroups of the group involved. In fact, Chelvam and Mutharasu (2013) published a paper containing subgroups of Z_n which are efficient dominating sets of a circulant graph for some generating subset. Chelvam and Kalaimurugan (2012) investigated some subgroups of the dihedral group which act as efficient dominating sets of a Cayley graph of this group. Balmaceda, Buloron, and Loquias (2015) classified all subgroups of the dicyclic group which are dominating sets with respect to a minimum inverse-symmetric generating subset. However, it was proven in the same paper that these Cayley graphs do not have any efficient dominating sets.

This article was inspired by the results in Chelvam and Kalaimurugan (2012) and Balmaceda et al. (2015). We determine all proper nontrivial subgroups of the dihedral group which dominate some Cayley graphs under a minimal inverse-symmetric generating subset, not containing the identity.

PRELIMINARIES

Let $\Gamma = (V, E)$ be an undirected graph with a vertex set V and edge set E. A subset D of V is called a dominating set of Γ if for every $x \in V \setminus D$, there exists $y \in D$ such that $\{x, y\} \in E$. The subset D is called *independent* if no two vertices in D are adjacent. Also, D is a *perfect dominating set* if each vertex outside D is adjacent to exactly one vertex in D. A dominating set D is *efficient* if it is both independent and perfect.

Let G be a group and $X \subseteq G$. The *Cayley graph* $Cay(G, X)$ is a directed graph with vertex set G and for any $a,b \in G$, (a,b) is an edge if and only if there exists $x \in X$ where $ax=b$. That is, the edges are

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obtained by right multiplication of elements of G by elements of X (Konstantinova, 2013). In order to obtain undirected edges, we let X be inverse-symmetric, i.e., if $x \in X$ then $x^{-1} \in X$. To avoid loops, X is assumed not to contain the identity element of G. If X generates G then $Cay(G,X)$ must be connected.

This paper focuses on some Cayley graphs of the *dihedral group* D_n where $n \geq 3$. This group can be described as a symmetry group of the regular n-gon consisting of rotations and reflections. We can also write $D_n = \langle r,s : r^n = 1 = s^2, srs = r^{-1} \rangle$ as a presentation of the dihedral group of order $2n$. Since we want subgroups of D_n which are dominating sets, the following listing of subgroups of the dihedral group by (Devi and Rajkumar, 2015) is helpful.

Proposition 1 Let $n \geq 3$. The following provides a complete list of subgroups of D_n:

i. cyclic subgroups $\langle r^d \rangle$, of order $\frac{n}{d}$, where d divides n;

ii. dihedral subgroups $\langle r^d, sr^i \rangle$, of order $\frac{2n}{d}$, where d divides n and $0 \leq i \leq d-1$.

RESULTS

This paper is directed towards proving the following main result.

Theorem 1 (Main Theorem)
Let $n \geq 4$ and $D_n = \langle r, s : r^n = 1 = s^2, srs = r^{-1} \rangle$. Let $S_t = \{r, r^{n-1}, sr^t\}$ where $t \in \{0,1,2,\ldots,n-1\}$. Then a proper nontrivial subgroup H of D_n is a dominating set of the Cayley graph $Cay(D_n, S_t)$ if and only if H is one of the following:

i. $\langle r \rangle$ for any n and for any t;
ii. $\langle r_2, sr^{2i+j} \rangle$ where 2 divides n, $j \in \{0,1\}$ and $t=2i$ or $t=2i+1$ for some $i \in \{0,1,\ldots,n-1\}$;
iii. $\langle r^3, sr^{2i+j} \rangle$ where 3 divides n, $j \in \{0,1,2\}$, and $t=2i$ or $t=2i+1$ or $2i \equiv t \pmod{n}$ for some i ;
iv. $\langle r^4, sr^{2i+2} \rangle$ where 4 divides n, $4<n$, $t=2i$ for some i;
v. $\langle r^4, sr^{2i+3} \rangle$ where 4 divides n, $4<n$, $t=2i+1$ for some i;
vi. $\langle sr^{2i+2} \rangle$ for $n=4$, $t=2i$ for some i;
vii. $\langle sr^{2i+3} \rangle$ for $n=4$, $t=2i+1$ for some i.

In order to derive such classification, we search for patterns among examples of Cayley graphs of specific dihedral groups. This means a series of tedious computations. We need to consider the following two cases.

The Case $X_n = \{r, r^{n-1}, s\}$

Let us start by determining what proper nontrivial subgroups of D_n dominate the Cayley graph $Cay(D_n, X_n)$. Observe that every proper nontrivial subgroup of D_3 forms a dominating set of $Cay(D_3, X_3)$.

The computational property in Lemma 1 is stated without proof and will be used vastly.

Lemma 1 Let $n \geq 4$ and D_n the dihedral group of order $2n$. Then $r^i s = sr^{n-i}$ for any $i \in \{0,1,2\ldots,n-1\}$.

Lemma 2 Let $n \geq 4$ and D_n the dihedral group of order $2n$. Let $H = \langle r \rangle$. Then the cosets of H form dominating sets of the Cayley graph $Cay(D_n, X_n)$.

Proof: Let $v \in D_n \backslash H$. Then $v = sr^i$ for some $i \in \{0,1,2,\ldots,n-1\}$. It follows that $vs = sr^i s = r^{n-i} \in H$ and so H is a dominating set of $Cay(D_n, X_n)$. Now, the only coset is sH. Suppose $v \in D_n \backslash sH$, then $v = r^i$ for some $i \in \{0,1,2,\ldots,n-1\}$ and $vs = r^i s = sr^{n-i} \in sH$

Hence, sH is also a dominating set of $Cay(D_n, X_n)$.

Lemma 3 Let n be divisible by 2 and $H = \langle r^2, sr^j \rangle$

where $j \in \{0,1\}$. Then the cosets of H form dominating sets of $Cay(D_n, X_n)$.

Proof: The proof follows similar computation as in the proof of Lemma 2.

Lemma 4 Let n be divisible by 3 and $H= \langle r^3, sr^j \rangle$ where $j \in \{0,1,2\}$. Then the cosets of H form dominating sets of $Cay(D_n, X_n)$.

Proof: This follows similar computation as in proof of Lemma 2.

Lemma 5 Let n be divisible by 4 and $H= \langle r^4, sr^2 \rangle$. Then the cosets of H form dominating sets of $Cay(D_n, X_n)$.

Proof: This follows similar computation as in the proof of Lemma 2.

Lemma 6 Let $n=4$ and $H=\langle sr^2 \rangle$. Then the cosets of H are dominating sets of $Cay(D_4, X_4)$.

Proof: This is a straightforward computation.

We remark that for $n=4$ and $i \in \{0,1,3\}$, $\langle sr^i \rangle$ is not a dominating set for $Cay(D_4, X_4)$. We extend this result to $n>4$. We first define the *closed neighborhood* of v in V as $N[v]=\{w \in V: \{v, w\} \in E\} \cup \{v\}$, for any graph (V, E).

Lemma 7 Let $n \geq 5$ and $H=\langle sr^i \rangle$. Then H is not a dominating set for $Cay(D_n, X_n)$.

Proof: If H is a dominating set of $Cay(D_n, X_n)$ then $D_n=N[sr^i] \cup N[1]$. However,

$|N[sr^i] \cup N[1]| \leq |N[sr^i]| + |N[1]| = 4+4 = 8 < 2(5) \leq 2n = |D_n|$.

Lemma 8 Let $n \geq 4$ and $d>1$ dividing n. Then $H=\langle r^d \rangle$ is not a dominating set of $Cay(D_n, X_n)$.

Proof: Consider sr^{n-d+1} which is not in $H= \langle r^d \rangle$. However:

1. $sr^{n-d+1} r = sr^{n-d+2} \notin H$;
2. $sr^{n-d+1} r^{n-1} = sr^{n-d} \notin H$;
3. $sr^{n-d+1} s = r^{d-1} \notin H$.

Hence, $\langle r^d \rangle$ does not dominate $Cay(D_n, X_n)$.

Lemma 9 Let n be divisible by 4 and $H=\langle r^4, sr^j \rangle$ with $j \in \{0,1,3\}$. Then H does not form a dominating set of $Cay(D_n, X_n)$.

Proof: Suppose H is a dominating set. Then there exists $x \in H$ adjacent to r^2. We have the following possibilities: $xr=r^2$ and so $x=r \notin H$ (impossible); $xr^{n-1}=r^2$ and so $x=r^3 \notin H$ (impossible); $xs=r^2$ and so $x=sr^{n-2}$ which means that $sr^2 \in H$. But this implies a contradiction. Thus, H cannot dominate r^2.

Lemma 10 Let n be divisible by $i \geq 5$. Then $H= \langle r^i, sr^j \rangle$ where $j \in \{0,1,\ldots,i-1\}$ is not a dominating set of $Cay(D_n, X_n)$.

Proof: Note that $r^2 r = r^3 \notin H$; $r^2 r^{n-1}) = r \notin H$; $r^3 r = r^4 \notin H$; $r^3 r^{n-1} = r^2 \notin H$. If r^2 is dominated by H then $r^2 s \in H$ and so $sr^{n-2} = r^2 s \in H$. Thus, $r^3 s = sr^{n-3} \notin H$. This means that H cannot dominate $Cay(D_n, X_n)$. Similar argument can be applied given that r^3 is dominated by H.

We now summarize these findings into the following theorem.

Theorem 2 Let $n \geq 4$ and $D_n = \langle r, s: r^n=1=s^2, srs=r^1 \rangle$. Denote $X_n=\{r, r^{n-1}, s\}$. Then a proper nontrivial subgroup H of D_n is a dominating set of the Cayley graph $Cay(D_n, X_n)$ if and only if H is one of the following:

i. $\langle r \rangle$ for any n;
ii. $\langle r^2, sr^j \rangle$ where $j \in \{0,1\}$ for any n divisible by 2;
iii. $\langle r^3, sr^j \rangle$ where $j \in \{0,1,2\}$ and any n divisible

by 3;
iv.$\langle r^4, sr^2\rangle$ for any n divisible by 4;
v.$\langle sr^2\rangle$ for $n=4$.

Proof: Lemmas 2 to 6 show that the subgroups on the list in the theorem are dominating sets of $Cay(D_n, X_n)$. On the other hand, lemmas 7 to 10 assure that there are no other subgroups of D_n which are dominating sets of $Cay(D_n, X_n)$.

The Case $Y_n = \{r, r^{n-1}, sr\}$

In this section, we only consider $n\geq 4$ which is even.

Lemma 11 Let $n\geq 4$ and D_n the dihedral group of order $2n$. Let $H=\langle r\rangle$. Then the cosets of H form dominating sets of the Cayley graph $Cay(D_n, Y_n)$.

Proof: Let $v \in D_n\backslash H$. Then $v=sr^i$ for some i $\in\{0,1,2,\ldots,n\text{-}1\}$. It follows that $vsr=sr^i sr=r^{n-i+1} \in H$. Suppose $v \in D_n\backslash sH$, then $\underline{v}=r^i$ for some i $\in \{0,1,2,\ldots,n\text{-}1\}$. Thus, $vsr=r^i sr=sr^{n-i+1} \in sH$. Hence, H and sH are dominating sets of $Cay(D_n, Y_n)$.

Lemma 12 Let $H=\langle r^2, sr^j\rangle$ where j $\in \{0,1\}$. Then the cosets of H form dominating sets of $Cay(D_n, Y_n)$.

Proof: The proof follows similar computation as in the proof of Lemma 11.

Lemma 13 Let n be divisible by 3 and $H=\langle r^3, sr^j\rangle$ where $j \in \{0,1,2\}$. Then the cosets of H form dominating sets of $Cay(D_n, Y_n)$.

Proof: The proof is similar to the proof of Lemma 11.

Lemma 14 Let n be divisible by 4 *and* $H=\langle r^4, sr^3\rangle$. Then the cosets of H form dominating sets of $Cay(D_n, Y_n)$.

Proof: This follows similar computation as in the proof of Lemma 11.

Lemma 15 Let $n=4$ and $H=\langle sr^3\rangle$. Then the cosets of H are dominating set of $Cay(D_4, Y_4)$.

Proof: This is a straightforward computation.

As above, it can be verified that for $n=4$ and $i \in \{0,1,2\}$,H$=\langle sr^i\rangle$ is not a dominating set of $Cay(D_4, Y_4)$.

Lemma 16 Let $n \geq 5$ and $H=\langle sr^i\rangle$. Then H is not a dominating set for $Cay(D_n, Y_n)$.

Proof: The proof follows similar argument with the proof of Lemma 7.

Lemma 17 Let $n \geq 4$ and $d>1$ dividing n. Then $H=\langle r^d\rangle$ is not a dominating set of $Cay(D_n, Y_n)$.

Proof: Similar computation is applied to sr^{n-d+2} as in the proof of Lemma 8.

Lemma 18 Let n be divisible by 4 and $H=\langle r^4, sr^j\rangle$ with $j \in\{0,1,2\}$. Then H does not form a dominating set of $Cay(D_n, Y_n)$.

Proof: The proof follows similar computation with the proof of Lemma 9.

Lemma 19 Let n be divisible by $i \geq 5$. Then $H=\langle r^i, sr^j\rangle$ where $j \in\{0,1,\ldots,i\text{-}1\}$ is not dominating set of $Cay(D_n, Y_n)$.
Proof: The proof is similar to the proof of Lemma 10.

We now collect the results of this section.

Theorem 3 Let $n \geq 4$ and n be even. Denote $Y_n = \{r, r^{n-1}, sr\}$. Then a proper nontrivial subgroup H of D_n is dominating set of the Cayley graph $Cay(D_n, Y_n)$ if and only if H is one of the following:

i. $\langle r \rangle$ for any n;
ii. $\langle r^2, sr^j \rangle$ where $j \in \{0,1\}$;
iii. $\langle r^3, sr^j \rangle$ where $j \in \{0,1,2\}$ for any n divisible by 3;
iv. $\langle r^4, sr^3 \rangle$ for n divisible by 4;
v. $\langle sr^3 \rangle$ for $n = 4$.

Proof: Lemmas 11 to 15 show that the subgroups on the list in the theorem are dominating sets of $Cay(D_n, Y_n)$. Meanwhile, lemmas 16 to 19 provide that no other subgroups of D_n are dominating sets of $Cay(D_n, Y_n)$.

CONCLUSION

Let us now present a proof of the main result of this paper as stated in Theorem 1.

Proof of Theorem 1

Let $n \geq 4$ be any integer and $t \in \{0,1,2,\ldots,n-1\}$. We have the following cases:

i. n is even, t is even:

This means that $t = 2i$ for some $i \in \{0,1,\ldots,\frac{n-2}{2}\}$. It can be observed that conjugation by r^i would give $X_n^{r^i} = S_t$. As in the proof of the main result of (Balmaceda et al, 2015), conjugation by r^i of the dominating sets in parts (*i*) to (*v*) of Theorem 2 would give all the possible dominating subgroups of $Cay(D_n, S_t)$.

ii. n is even, t is odd:

This implies that $t = 2i+1$ where $i \in \{0,1,\ldots,\frac{n-2}{2}\}$. Computation shows that $Y_n^{r^i} = S_t$ and conjugation by r^i of the dominating sets in parts (*i*) to (*v*) of Theorem 3 would give all the possible dominating subgroups.

iii. n is odd:

If $t = 2i$ for some $i \in \{0,1,\ldots,\frac{n-2}{2}\}$, then $X_n^{r^i} = S_t$ and conjugation by r^i to parts (*i*) and (*iii*) of Theorem 2 gives us the desired dominating sets. If t is odd then there exists $i \in \{0,1,\ldots, n-1\}$ such that $t \equiv 2i \pmod{n}$, since n is odd. As above, we obtain $X_n^{r^i} = S_t$ and by conjugating parts (*i*) and (*iii*) of Theorem 2, the dominating subgroups for $Cay(D_n, S_t)$ are determined.

We would like to remark that unlike the dicyclic group case as in Balmaceda et al. (2015), the subgroups $\langle r^4, sr^{2i+2} \rangle$ and $\langle r^4, sr^{2i+3} \rangle$ above are efficient dominating sets for their corresponding Cayley graphs.

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